12.2 PROPOSITIONAL CALCULUS

Propositional calculus also called *statement calculus* is the branch of mathematics that is used to describe a logical system. A logical system or structure consists of: (*i*) a universe of statements, (*ii*) truth tables as axioms for the logical operators, and (*iii*) definitions that explain equivalence and implication of statements.

A proposition (or statement) is used to describe any mathematical structure. It is a declarative sentence that is either *true* or *false*, but not both. However, a statement becomes *predicate* when variables are assigned an appropriate type of values in a formula to drawn conclusions. Questions, exclamations and commands are not propositions.

The following statements are of declarative type:

(*i*) $\{2, 3\} \subseteq \{1, 2, 3, 4\}$

(*ii*) $a^2 + b^2 = (a + b)^2 - 2ab$, for all values of a and b

(*ii*) 9 is greater that 7.

(*iv*) The earth is round.

(*v*) The price of wheat in Delhi rose on 15 August 1998.

(*vi*) Jaipur is the capital of Rajasthan.

However, following are not the statements:

(*i*) What is the exchange rate of Indian rupees to US dollars?

(*ii*) What are you doing?

(*iii*) May God bless you?

(*iv*) Who are you?

(*v*) Lahore is in Bangladesh.

12.2.1 Propositional Variables and Constants

For convenience English alphabet: $p, q, r, s, \ldots$, etc., are used to represent simple statements and are called propositional variables. For example, we can write,

$$p = \text{It is raining}, \quad q = \text{It is cold.}$$

Similarly, two letters T and F that are used to represent true and false statement are called *propositional constants*.

12.2.2 Logical Connectives and Compound Proposition

Fundamental connectives Two or more simple propositions (or statements) can be combined by means of logical operators or connectives to form a single proposition called *compound proposition* (*composite* or *molecular*). However, a proposition consisting of only one propositional variable (or constant) is called *atomic* (*primary* or *primitive*) proposition.

The words and phrases (or symbols) that are used to form a compound proposition are called *connectives*. There are five logical connectives which are frequently used for this purpose. These are shown in Table 12.1.

Table 12.1 Logical Connectives

Symbol	*Connective Word*	*Name*
$\sim$	not	negation
$\wedge$	and	conjunction
$\vee$	or	disjunction
$\Rightarrow$	implies or If . . . then	implication or conditional
$\Leftrightarrow$	If and only if	equivalence or biconditional

If p and q are any two propositions, then $\sim q$, $p \wedge q$, $p \vee q$, $p \Rightarrow$, $p \Leftrightarrow q$ $\{\equiv (p \Rightarrow q) \wedge (q \Rightarrow p)\}$ are all propositions.

A compound proposition (or statement) which does not contain any connective is called a *prime proposition*. Few propositions are composed of several simple propositions and their true value is determined by the truth value of each simple propositions and the type of connectives used. The meaning of the connectives can be summarised in the tabular form known as *truth tables*.

The compound proposition can be constructed as: $p \wedge q \wedge r$, $p \wedge (q \Rightarrow r)$, etc. The precedence conventions are in the order given above so that $\sim p_1 \wedge p_2 \vee p_3 \Rightarrow p_4 \Leftrightarrow p_5$ is taken as: $((((\sim p_1) \wedge p_2) \vee p_3) \Rightarrow p_4) \Leftrightarrow p_5$. However, it is good to insert brackets even where it is unneccessary, if this improves readability.

Truth tables A truth table is the convenient way of summarising the truth values of logical statements. It indicates the truth values of compound propositions composed of several simple propositions. A truth table consists of columns and rows. The number of columns depend upon the numbers of sample propositions and connectives used to form a compound proposition. The number of rows in a truth table are found on the basis of simple propositions. For example, if there are two simple propositions, there will be $2^2 = 4$ rows. In general, for n simple propositions, the total number of rows will be 2^n.

Since each proposition (or statement) has two possible truth values, we assign letter T to a proposition when it is true and the letter F when it is false. The truth table is useful in

(*i*) finding out the validity of an equivalence relation between functions.

(*ii*) designing and testing the electronic circuits to perform a given operation based on certain relationship.

12.3 BASIC LOGICAL OPERATIONS

As mentioned earlier that there are three basic logical operations, namely, *conjunction*, *disjuction* and *negation*, which correspond to the English words *and*, *or* and *not*, respectively. These logical operations are discussed in detail below.

12.3.1 Conjunction

When two or more statements are joined by the connective, denoted by the symbol, $\wedge$, the compound statement so formed is called a *conjunction*. Let p and q be two statements. Then $p \wedge q$ form a statement which is true if and only if both p and q are true, and is false if p is false or q is false or both are false. The statement $p \wedge q$ is read as: *p and q*, and is called the *conjunction of p and q*.

For example, let p be the statement 'Delhi is the capital of India' and q be the statement '8 + 5 = 13'. Then $p \wedge q$ which is read as 'Delhi is the capital of India and 8 + 5 = 13' is true. If, however, q is the statement '8 + 5 = 12', then $p \wedge q$ is false even though p is true.

Truth values of the statement $p \wedge q$ in terms of truth values of p and q are given in the truth table shown in Table 12.2.

Table 12.2 Truth Table for $p \wedge q$

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

From Table 12.2, the following observations may be noted:

(*i*) With two statements, there are $2^2 = 4$ possible combinations.

(*ii*) The basic truth values in the first two columns are same as binary digits if you place 1 for T and 0 for F.

Remark In logic we may join two totally unrelated statements by the connective '*and*'. For example, the statement $p \wedge q$: Raju is handsome and Ramesh is selfish, is obtained by joining two statements p: Raju is handsome and q: Ramesh is selfish, which are not related.

12.3.2 Disjunction

When two or more statements are joined by the connective *or* denoted by the symbol, $\vee$, the compound statement, so formed is called a *disjunction* of two statements p and q and is written as p or q or $p \vee q$.

The connective has two different meanings: *Inclusive Disjunction* and *Exclusive Disjunction*. In the inclusive sense the compound statement $p \vee q$ is true only when at least one of the statements is true. But in the exclusive sense the compound statement $p \vee q$ is true only when either p or q but not both, is true.

The exclusive disjunction is commonly called *inequivalence* (written as $p \not\equiv q$), since it defines a statement that is true precisely when p and q have opposite or equivalent truth values. Two statements p and q are logically equivalent ($\equiv$) if they have same truth values.

Illustrations

1. Let p : 5 is a positive integer, and

   q : $\sqrt{5}$ is a rational number.

   Then $p \vee q$: '5 is a positive integer' or '$\sqrt{5}$ is a rational number' is the true disjunction because p is true even though q is false.

2. Let p : Ganga is the mountain, and

   q : Delhi is the capital of England.

   Then $p \wedge q$: Ganga is the mountain or Delhi is the capital of England is false disjunction because both p and q are false.

3. Let p : XYZ company earned 20 per cent profit per share in 1998.

   q : XYZ company paid 12 per cent dividend per share in 1998.

   The inclusive disjunction of p and q is

   $p \vee q$: XYZ company earned 20 per cent profit per share in 1998 or XYZ company paid 12 per cent dividend per share in 1998 or both.

   The exclusive disjunction of p and q is

   $p \vee q$: XYZ company earned 20 per cent profit per share in 1998 or XYZ company paid 12 per cent dividend per share in 1998 but not both.

4. In the compound statement, p : This weekend I will go to Shimla or Nainital, the use of 'or' is not clear. Because, if 'or' means inclusive disjunction, then at least one or possibly both cities will be visited. If 'or' means exclusive disjunction, the only one city will be visited.

   If p and q are two simple statements, then truth values of and $p \wedge q$ and $p \vee q$ are given in the truth table shown in Tables 12.3(a) and (b).

Table 12.3(a) Truth Table for $p \wedge q$

p	q	$p \wedge q$
T	T	T
F	T	F
T	F	F
F	F	T

Table 12.3(b) Truth Table for $p \vee q$

p	q	$p \vee q$
T	T	T
F	T	T
T	F	T
F	F	F

In table 12.3(b), it may be noted that the statement $p \vee q$ is true if either p or q or both are true. If both p and q are false, the statement $p \vee q$ is false. The statement $p \wedge q$ is true only when both p and q are either true or false.

12.3.3 Negation

If p is a statement, then *negation* (*or not*) of p, denoted by $\sim p$ is defined to form a statement that is true when p is false, and false when p is true. The negation of statement p is read as 'not p'.

In fact, ~ (not) is not a connective because it does connect two statements. However, ~ p is a statement if p is the statement.

Illustrations

1. If $p : x = 3$, then $\sim p : x \neq 3$.
2. If p : Sun is shinning, then ~ p : sun is not shinning

The truth values of ~ p relative to p are given in Table 12.4.

Table 12.4 Truth Table for Negation (~)

p	$\sim p$
T	F
F	T

Double negative is positive and can be justified from the truth Table 12.4.

Table 12.5 True Table for Double Negation [~ (~)]

p	$\sim p$	$\sim(\sim p)$
T	F	T
F	T	F

In Table 12.5 it may be noted that double negation of any statement gives the original statement.

Negation of compound statements When a compound statement is regreted, its logical connective changes from *and* to *or* and from *or* to *and*. For example,

$$\sim(p \wedge q) = \sim p \vee \sim q$$

$$\sim(p \vee q) = \sim p \wedge \sim q$$

The following relations are based on above laws:

(*a*) $\sim(p \wedge \sim q) = \sim p \vee \sim(\sim q) = \ \sim p \vee q$

(*b*) $\sim(\sim p \wedge q) = \sim(\sim q) \vee \sim q = p \vee \sim q$

(*c*) $\sim(\sim p \vee \sim q) = \sim(\sim p) \wedge \sim(\sim q) = p \wedge q$

Example 1 Let p be the statement 'South-West monsoon is very good this year' and q be the statement 'Rivers are rising'. Give the verbal translations of statement (*a*) and verify the statement (*b*).

(*a*) (*i*) $p \vee \sim q$ and (*ii*) $\sim(\sim p \vee \sim q)$.

(*b*) statement $x > 1$ is false if and only if $x^2 > 1$ is false, when x is a real number.

Solution (*a*) (*i*) The South-West monsoon is very good this year or the rivers are not rising.

(*ii*) It is not true that the South-West monsoon is not very good this year and the rivers are not rising. In other words, we can also say that the South-West monsoon is very good this year but the rivers are rising.

(*b*) The statement $x > 1 \Leftrightarrow x^2 > 1$ is false because if $x^2 > 1$ means $x > 1$ or $x < 1$.

Example 2 With the help of truth tables, prove that

(a) $\sim p \vee \sim q = \sim (p \wedge q)$

(b) $\sim (p \vee q) = \sim (p \wedge \sim q)$

(c) $p \vee q = \sim (\sim p \wedge \sim q)$

(d) $p \vee \sim q = (p \vee q) \wedge \sim (p \wedge q)$

(e) $\sim (p \wedge q) = \sim p \vee \sim q$

Solution (a)

Table 12.6 Truth Table

p (1)	q (2)	$\sim p$ (3)	$\sim q$ (4)	$\sim p \vee \sim q$ (5)	$p \wedge q$ (6)	$\sim (p \wedge q)$ (7)
T	T	F	F	F	F	F
T	F	F	T	T	F	T
F	T	T	F	T	F	T
F	F	T	T	T	F	T

Since values in Columns (5) and (7) are identical, therefore, $\sim p \vee \sim q = \sim (p \wedge q)$.

(b)

Table 12.7 Truth Table

p (1)	q (2)	$p \vee q$ (3)	$\sim (p \vee q)$ (4)	$\sim p$ (5)	$\sim q$ (6)	$\sim p \wedge \sim q$ (7)
T	T	T	F	F	F	F
T	F	T	F	F	T	F
F	T	T	F	T	F	F
F	F	F	T	T	T	T

Since values in Columns (4) and (7) are identical, therefore, $\sim (p \vee q) = \sim p \wedge \sim q$.

(c)

Table 12.8 Truth Table

p (1)	q (2)	$p \vee q$ (3)	$\sim p$ (4)	$\sim q$ (5)	$\sim p \wedge \sim q$ (6)	$\sim(\sim p \wedge \sim q)$ (7)
T	T	T	F	F	F	T
T	F	T	F	T	F	T
F	T	T	T	F	F	T
F	F	F	T	T	T	F

Since values in Columns (3) and (7) identical, therefore, $p \vee q = \sim (\sim p \wedge \sim q)$.

(d)

Table 12.9 Truth Table

p (1)	q (2)	$p \vee \sim q$ (3)	$p \vee q$ (4)	$p \wedge q$ (5)	$\sim (p \vee q)$ (6)	$(p \vee q) \wedge \sim (p \wedge q)$ (7)
T	T	F	T	T	F	F
T	F	T	T	F	T	T
F	T	T	T	F	T	T
F	F	F	F	F	T	F

Since values in Columns (3) and (7) are identical, therefore, $p \vee \sim q = (p \vee q) \wedge \sim (p \wedge q)$.

12.4 STATEMENTS GENERATED BY A SET

Let $X = \{p, q, r\}$ be a finite set of statements p, q and r. The following are two definitions for a compound statement generated by X.

(*i*) A compound statement generated by X is any valid combination of statements p, q and r in X with logical operators: conjunction, disjunction and negation.

(*ii*) (*a*) $p \in X$ be any arbitrary element. Then p is said to be the statement generated by X.

(*b*) If A and B are two statements generated by X, then ~ A, A ∧ B, A ∨ B, etc., are also statements.

As mentioned above, the precedence conventions for interpreting statements are in the order: *negation*, *conjunction* and *disjunction*. For example, instead of writting $(p \wedge q) \wedge r$, we may write $p \wedge q \vee r$. Some of the expressions and their parenthesized versions are:

(*i*) $((p \wedge (q)) \wedge r$ for $p \wedge q \wedge r$

(*ii*) $(\sim p) \vee (\sim r)$ for $\sim p \vee \sim r$

(*iii*) $\sim (\sim p)$ for $\sim \sim p$

The truth values of any statement, $c : (p \wedge q) \vee (\sim q \wedge r)$ generated by the combination of statements p, q are r with logical connectives can be obtained as shown in Table 12.20.

Table 12.20 Truth Table for $(p \wedge q) \vee (\sim q \wedge r)$

p (1)	q (2)	r (3)	$(p \wedge q)$ (4)	$\sim q$ (5)	$(\sim q \wedge r)$ (6)	$(p \wedge q) \vee (\sim q \wedge r)$ (7)
F	F	F	F	T	F	F
F	F	T	F	T	T	T
F	T	F	F	F	F	F
F	T	T	F	F	F	F
T	F	F	F	T	F	F
T	F	T	F	T	T	T
T	T	F	T	F	F	F
T	T	T	T	F	F	F

Remarks 1. Conditional (⇒) and biconditional (⇔) connectives have not been included in the definition of a statement generated by a set because these can be obtained from three main connectives: conjunction, disjunction and negation.

2. A compound statement in its expression generated by a set need not include its each element. For example, in Table 12.20 the truth value of the statement $\sim q \wedge r$ is same as the truth values of the statement $(p \wedge q) \vee (\sim q \wedge r)$ generated by p, q and r.

12.5 CONDITIONAL STATEMENTS

Statements such the following occur quite frequently in mathematics and computer science, and therefore, an understanding of their nature is very important.

(*i*) If you read, then you will pass the examination.

(*ii*) If you do your homework, then you will be allowed to play.

(*iii*) If $9 + 3 = 12$, then $12 - 9 = 3$.

(*iv*) If it rains, then there will be no picnic.

In all these statements, two simple statements are connected by *If . . ., then* to form a single statement. In other words, all such statements can be restated to fit into the form *if* (*condition* . . .), *then* (*conclusion*). Therefore, a compound statement using the connective 'If . . ., then' is called a *conditional statement* or *implication*.

A conditional statement is used as a guarantee for the conclusion to be true provided condition is true. If p and q are two statements, then compound statement *if p, then q* denoted by $p \Rightarrow q$ is an implication, and the connective *if . . ., then . . .,* is the *conditional connective*.

The statement, p is called the *hypothesis* or *antecedent*. The hypothesis p is often a relational expression such as $x > 5$ and becomes a logical statement that has the truth value 0 or 1, depending on the value of the variable x. The statement q is called the *consequent* or *conclusion*, i.e. it is an 'executable statement'. While dealing with 'if p and then q', the computer executes q and on the condition that p is true. Otherwise computer goes to the next instruction in the program sequence. For example, for the decision structure 'if p then q else r', q is executed when p is true and r is executed when p is false. The connective *if . . ., then* is denoted by $\rightarrow$ or $\Rightarrow$ and can be read as follows:

(*i*) p implies q

(*ii*) p is sufficient for q

(*iii*) p only if q

(*iv*) q is necessary for p

(*v*) q is consequence of p

Remark Reader should not confuse with $p \Rightarrow q$ and $q \Rightarrow p$. These two conditions are completely different. The statement $p \Rightarrow q$ is ture when either both p and q are true or false expect when p is true and q is false.

Truth Table for $p \Rightarrow q$

The truth values of $p \Rightarrow q$ in terms of the truth values of p and q are shown in Table 12.21.

Table 12.21 Truth Table for $p \Rightarrow q$

p	q	$p \Rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

Example 7 Write each of the following as conditional sentences in the *If . . ., then* form:

(*i*) Freezing water expands.

(*ii*) A racer wins the race only if he runs fast.

(*iii*) Roses are vegetables if carrots are flowers.

Solution (*i*) If water freezes, then it expands.

(*ii*) If a racer wins the race, then he runs fast.

(*iii*) If carrots are flowers, then roses are vegetables.

12.6 CONVERSE, INVERSE AND CONTRAPOSITIVE STATEMENTS

12.6.1 Converse Statements

The converse of a given conditional statement, $p \Rightarrow q$ is a new statement formed by interchanging the phypothesis, *p* and the *conclusion*, *q* of the given conditional statement. For example, if $p \Rightarrow q$, then converse of it is $q \Rightarrow p$.

Illustration Consider the statements:

p : You are guilty, q : You are punished

The implication $p \Rightarrow q$ states that *If you are guilty, then you are punished.* The converse of this implication, $q \Rightarrow p$ states that *If you are punished, then you are guilty.*

Obviously, the statement: *all guilty persons are punished*, is not the same as the statement: *every one who is punished is also guilty.* The truth values for $p \Rightarrow q$ and its converse are gives in Table 12.22.

Table 12.22 Truth Table of Converse

p	q	$p \Rightarrow q$	$q \Rightarrow p$
T	T	T	T
T	F	F	T
F	T	T	F
F	F	T	T

12.6.2 Inverse Statements

The inverse of a given conditional statement, $p \Rightarrow q$ is a new statement whose hypothesis is the negation of the original hypothesis and whose conclusion is the negation of the original conclusion. For example, if $p \Rightarrow q$, then inverse of it is *if not p, then not q* and is written as $\sim p \Rightarrow \sim q$.

Illustration As mentioned above, the statement $p \Rightarrow q$ means that 'If you are guilty, then you are punished.' The inverse of this statement is $\sim p \Rightarrow \sim q$ which means that 'If you are not a guilty, then you are not punished.' The truth values for $p \Rightarrow q$ and its inverse are given in Table 12.23.

Table 12.23 Truth Table of Inverse

p	q	$\sim p$	$\sim q$	$p \Rightarrow q$	$\sim p \Rightarrow \sim q$
T	T	F	F	T	T
T	F	F	T	F	T
F	T	T	F	T	F
F	F	T	T	T	T

12.6.3 Contrapositive Statements

The contrapositive of a given conditional statement, $p \Rightarrow q$ is a new statement whose conclusion is the negation of the original hypothesis. For example, if $p \Rightarrow q$, then contrapositive of it is *if not q, then not p*, and is written as $\sim q \Rightarrow \sim p$.

Illustration As mentioned above, the statement, $p \Rightarrow q$ means that 'If you are guilty, then you are punished'. The contrapositive of this statement is $\sim q \Rightarrow \sim p$, which means that 'If you are not punished, then you are not guilty.' The truth values for $p \Rightarrow q$ and its contrapositive are given is Table 12.24.

Table 12.24 Truth Table of Contrapositive

p	q	$\sim p$	$\sim q$	$p \Rightarrow q$	$\sim q \Rightarrow \sim p$
T	T	F	F	T	T
T	F	F	T	F	F
F	T	T	F	T	T
F	F	T	T	T	T

In Table 12.24 it may noted that the entries under $p \Rightarrow q$ and contrapositive are same. Thus, the statement $p \Rightarrow q$ is logically equivalent to its contrapositive '$\sim q \Rightarrow \sim p$'. Also from Tables 12.22 and 12.23, it may be noted that the converse $q \Rightarrow p$ of conditional statement $p \Rightarrow q$ is logically equivalent to its inverse $\sim p \Rightarrow \sim q$. Hence, we conclude that

(*i*) $p \Rightarrow q \equiv \sim q \Rightarrow \sim p$

(*ii*) $q \Rightarrow p \equiv \sim p \Rightarrow \sim q$

In other words, given any simple conditional statement, we may obtain the following:

Statement : If $p \Rightarrow q$ is an implication

Converse : then converse of $p \Rightarrow q$ is the implication $q \Rightarrow p$, but $q \Rightarrow p \neq p \Rightarrow q$

Inverse : then inverse of $p \Rightarrow q$ is the implication $\sim p \Rightarrow \sim q$, but $\sim p \Rightarrow \sim q \neq p \Rightarrow q$

Contrapositive : then contrapositive of $p \Rightarrow q$ is the implication, $\sim q \Rightarrow \sim p$

The summary of the above definitions is given in truth Table 12.25.

Table 12.25 Summary of Converse, Inverse and Contrapositive

p	q	*Conditional* $p \Rightarrow q$	*Converse* $q \Rightarrow p$	*Inverse* $\sim p \Rightarrow \sim q$	*Contrapositive* $\sim q \Rightarrow \sim p$
T	T	T	T	T	T
T	F	F	T	T	F
F	T	T	F	F	T
F	F	T	T	T	T

Example 8 Prove that $p \Rightarrow q \equiv \sim q \Rightarrow \sim p$.

Solution Truth values for the given result is shown in Table 12.26.

Table 12.26 $p \Rightarrow q \equiv \sim q \Rightarrow \sim p$

p (1)	q (2)	$p \Rightarrow q$ (3)	$\sim q$ (4)	$\sim p$ (5)	$\sim q \Rightarrow \sim p$ (6)
T	T	T	F	F	T
T	F	F	T	F	F
F	T	T	F	T	T
F	F	T	T	T	T

Since entries in Columns (3) and (6) are identical, therefore, $p \Rightarrow q = \sim q \Rightarrow \sim p$.

Example 9 Prove by means of a truth table that $\sim (p \Rightarrow q) = p \wedge \sim q$.

Solution Truth values for the given result is shown in Table 12.27.

Table 12.27 $\sim (p \Rightarrow q) \equiv p \wedge \sim q$

p (1)	q (2)	$p \to q$ (3)	$\sim (p \to q)$ (4)	$\sim q$ (5)	$p \wedge \sim q$ (6)
T	T	T	F	F	F
T	F	F	T	T	T
F	T	T	F	F	F
F	F	T	F	T	F

Since entries in Columns (4) and (6) are identical, therefore, $\sim (p \Rightarrow q) = p \wedge \sim q$.

Example 10 Prove that conditional operation distributes over conjunction, i.e.

$$p \Rightarrow (q \wedge r) \equiv (p \Rightarrow q) \wedge (p \Rightarrow r)$$

Solution Truth values for the given result is shown in Table 12.28.

Table 12.28 $(q \wedge r) \equiv (p \Rightarrow q) \wedge (p \Rightarrow r)$

p (1)	q (2)	r (3)	$q \wedge r$ (4)	$p \Rightarrow (q \wedge r)$ (5)	$p \Rightarrow q$ (6)	$p \Rightarrow r$ (7)	$(p \Rightarrow q) \wedge (p \Rightarrow r)$ (8)
T	T	T	T	T	T	T	T
T	F	T	F	F	F	T	F
F	T	T	T	T	T	T	T
F	F	T	F	T	T	T	T
T	T	F	F	F	T	F	F
T	F	F	F	F	F	F	F
F	T	F	F	T	T	T	T
F	F	F	F	T	T	T	T

Since entries in Columns (5) and (8) are identical, therefore, the result is true.

Example 11 If p, q and r are three statements, then show that $[(p \Rightarrow q) \wedge (q \Rightarrow r)] \Rightarrow (p \Rightarrow r)$. Also write its truth set. *[UPTU, MCA, 2008]*

Solution Truth values for the given result is shown in Table 12.29.

Table 12.29 $[(p \Rightarrow q) \wedge (q \Rightarrow r)] \Rightarrow (p \Rightarrow r)$

p (1)	q (2)	r (3)	$p \Rightarrow q$ (4)	$q \Rightarrow r$ (5)	$(p \Rightarrow q) \wedge (q \Rightarrow r)$ (6)	$p \Rightarrow r$ (7)	(6) $\Rightarrow$ (7) (8)
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	T
T	F	T	F	T	T	F	T
T	F	F	F	T	F	F	T
F	T	T	T	T	T	T	T
F	T	F	T	F	T	F	T
F	F	T	T	T	T	T	T
F	F	F	T	T	T	T	T

Since entries in Column (9) are Ts only, therefore, the given statement is true or valid.

Example 14 Giving reasons in support of your answers decide if the two compound statements given below are equivalent statements:

(*a*) If Ram is 21 years old, then he has right to vote.

(*b*) Ram is not 21 years old, then he has a right to vote.

Solution Let p be the statement *Ram is 21 years old* and q is the statement, *he has a right to vote.* Then statement (*a*) is of the form $p \Rightarrow q$ and statement (*b*) is of the form $\sim p \vee q$. Now in order to show that the two compound statements are equivalent, we have to show that biconditional formed by them is true for all assignments of p and q. Truth values for the compound statement are shown in Table 12.32.

Table 12.32 Truth Table for $(p \Rightarrow q) \Leftrightarrow (\sim p \vee q)$

p (1)	$\sim p$ (2)	q (3)	$p \Rightarrow q$ (4)	$\sim p \vee q$ (5)	$p \Rightarrow q \Leftrightarrow \sim p \vee q$ (6)
T	F	T	T	T	T
T	F	F	F	F	T
F	T	T	T	T	T
F	T	F	T	T	T

Since entries in Column (6) are only Ts, therefore, the biconditional formed by p and q is true for all possible assignments. Hence, the two given compound statements are equivalent.

Example 15 Prove that $\sim (p \Rightarrow q) \equiv p \cap \sim q$.

Solution The truth values for the given result is shown in Table 12.33.

Table 12.33 $\sim (p \Rightarrow q) \equiv p \cap - q$

p (1)	q (2)	$p \rightarrow q$ (3)	$\sim (p \rightarrow q)$ (4)	$\sim q$ (5)	$p \cap \sim q$ (6)
T	T	T	F	F	F
T	F	F	T	T	T
F	T	T	F	F	F
F	F	T	F	T	F

Since entries in Columns (4) and (6) are identical, therefore, the given statement is true.

Example 16 Construct truth tables for the following statements:

(*a*) $(p \Rightarrow p) \Rightarrow (p \Rightarrow \sim p)$

(*b*) $(p \Rightarrow p) \vee (p \Rightarrow \sim p)$

(*c*) $(\sim q \Rightarrow \sim p) \Rightarrow (p \Rightarrow q)$

(*d*) $(p \Rightarrow (q \Rightarrow r)) \Rightarrow ((p \Rightarrow q) \Rightarrow (p \Rightarrow r))$

Solution (*a*) Truth values for the statement are shown in Table 12.34.

Table 12.34 $(p \Rightarrow p) \Rightarrow (p \Rightarrow \sim p)$

p	$\sim p$	$p \Rightarrow p$	$p \Rightarrow \sim p$	$(p \Rightarrow p) \Rightarrow (p \Rightarrow \sim p)$
T	F	T	F	F
F	T	T	T	T

Example 20 Construct the truth table for the following and write the truth set

$$p \Rightarrow [(p \vee r) \wedge \sim (p \Leftrightarrow \sim r)].$$

Solution Truth values for the given statement is shown in Table 12.48.

Table 12.48 $p \Rightarrow [(p \vee r) \wedge \sim (p \Leftrightarrow \sim r)]$

p (1)	q (2)	r (3)	$\sim r$ (4)	$q \vee r$ (5)	$p \Leftrightarrow \sim r$ (6)	$\sim (p \Leftrightarrow \sim r)$ (7)	$(q \vee r) \wedge \sim (p \Leftrightarrow r)$ (8)	$p \Rightarrow [(q \vee r) \wedge \sim (p \Leftrightarrow \sim r)$ (9)
T	T	T	F	T	F	T	T	T
T	T	F	T	T	T	F	F	F
T	F	T	F	T	F	T	T	T
T	F	F	T	F	T	F	F	F
F	T	T	F	T	T	F	F	T
F	T	F	T	T	F	T	T	T
F	F	T	F	T	T	F	F	T
F	F	F	T	F	F	T	F	T

The required truth set is: {TTT, TFT, FTT, FTF, FFT, FFF}. The elements of the set are the values of p, q and r, respectively for T values in Column (8).

12.8 TAUTOLOGIES

Earlier in this chapter we have seen how compound statements are obtained from given simple statements by using connectives. Ordinarily when a compound statement is written, the truth of such statement is not known unless we know the truth or falsity of the component statements. There are certain compound statements that always are true regardless of the truth or falsity of component statements. Compound statements which have this characteristic are called *tautologies*. In other words, *tautology is a compound statement* if it is true for all truth value assignments for its component statements. Obviously, the truth table of a tautology will contain only T entries in the last column.

Importance of tautology Tautology helps conclude some statement from some given statements. For example, if the statement $p \Rightarrow q$ is a tautology, then it is easy to conclude the truth of q from the truth of p. Thus with the help of tautology we move from some given statement to some concluding statement in a step-by-step manner which is justified within the framework of mathematical logic.

Example 21 Show that the statement $(p \wedge q) \Rightarrow q$ is a tautology.

Solution The truth table for the given statement is shown in Table 12.49.

Table 12.49 $(p \wedge q) \Rightarrow q$

p	q	$p \wedge q$	$p \wedge q \Rightarrow q$
T	T	T	T
T	F	F	T
F	T	F	T
F	F	F	T

Since the statement $(p \wedge q) \Rightarrow q$ has its truth value T for all its entries in the truth table. Hence, it is a tautology.

Example 22 Show that the statement $p \vee \sim p$ is a tautology.

Solution The truth table for the given statement is shown in Table 12.50.

Table 12.50 $p \vee \sim p$

p	$\sim p$	$p \vee \sim p$
T	F	T
F	T	T

Since the statement $p \vee \sim p$ has its truth value T for all its entries in the table, therefore, it is tautology. This is called the *law of exclusive middle,* i.e. either p is true or it is false. There is no middle possibility.

12.9 CONTRADICTION

If a compound statement is false for all true value assignments for its component statements, then it is called a *contradiction*, i.e. a compound statement is said to be contradicting its truth value if it is false (F) for all its entries in the truth table. If a statement is a contradiction, then its negation will be a tautology.

Example 23 Show that the statement $p \wedge \sim p$ is a tautology or contradiction.

Solution The truth table for $p \wedge \sim p$ is shown in Table 12.51.

Table 12.51 $p \wedge \sim p$

p	$\sim p$	$p \wedge \sim p$
T	F	F
F	T	F

Since the statement $p \wedge \sim p$ has its false value F for all its entries in the truth table. Hence, it is a tautology.

Example 24 Establish $\sim(p \wedge q) \Leftrightarrow (\sim p \vee \sim q)$ as a tautology.

Solution The truth table for the given statement is shown in Table 12.52.

Table 12.52 $\sim(p \wedge q) \Leftrightarrow (\sim p \vee \sim q)$

p (1)	q (2)	$p \wedge q$ (3)	$\sim(p \wedge q)$ (4)	$\sim p$ (5)	$\sim q$ (6)	$\sim p \vee \sim q$ (7)	$\sim(p \wedge q) \Leftrightarrow (\sim p \vee \sim q)$ (8)
T	T	T	F	F	F	F	T
T	F	F	T	F	T	T	T
F	T	F	T	T	F	T	T
F	F	F	T	T	T	T	T

Since Column (8) contains only Ts, therefore, the given proposition is a tautology.

Example 25 Show that the statement $(p \wedge q) \Rightarrow (p \vee q)$ is a tautology, but $(p \vee q) \Rightarrow (p \wedge q)$ is not.

Solution Truth table for $(p \vee q) \Rightarrow (p \wedge q)$ is shown in Table 12.53.

Table 12.53 $(p \vee q) \Rightarrow (p \wedge q)$

p	q	$p \vee q$	$p \wedge q$	$(p \vee q) \Rightarrow (p \wedge q)$
T	T	T	T	T
T	F	T	F	F
F	T	T	F	F
F	F	F	F	T

Since, the given statement does not have all entries as T values in the last column of the truth table, therefore it is not a tautology.

Example 26 By means of a truth table, show that the statement $p \vee \sim(p \wedge q)$ is a tautology.

Solution The truth table for the given statement is shown in Table 12.54.

Table 12.54 $p \vee \sim(p \wedge q)$

p	q	$p \wedge q$	$\sim(p \wedge q)$	$p \vee \sim(p \wedge q)$
T	T			
T	F	T		
T	F	F	T	T
F	T	F	T	T
F	F	F	T	T

Since the given statement has its truth values T for all its entries in the last column of the truth table, therefore, it is a tautology.

Example 27 Show that $(p \wedge q) \wedge \sim(p \vee q)$ is a fallacy or contradiction.

Solution Truth table for the given statement is shown in Table 12.55.

Table 12.55 $(p \wedge q) \wedge \sim(p \vee q)$

p (1)	q (2)	$p \wedge q$ (3)	$p \vee q$ (4)	$\sim(p \vee q)$ (5)	$(p \wedge q) \wedge \sim(p \vee q)$ (6)
T	T	T	T	F	F
T	F	F	T	F	F
F	T	F	T	F	F
F	F	F	F	T	F

Since the statement has its false values (F) for all its entries in the last column of the truth table, therefore, it is a fallacy.

12.10 CONTINGENCY

Contingency is a compound statement whose truth or falsity depends upon the truth values of its component statements.

Example 28 Show that the following statements are contingency.

(a) $p \Rightarrow (p \Rightarrow q)$ (b) $(p \wedge \sim q) \vee (\sim p \wedge q)$

(c) $\sim (p \vee q) \wedge (\sim p \vee \sim q)$ (d) $(p \Rightarrow (q \wedge r)) \Rightarrow \sim (p \Rightarrow q)$

Solution (a) Truth table for the given statement is shown in Table 12.56.

Table 12.56 $p \Rightarrow (p \Rightarrow q)$

p	q	$p \Rightarrow q$	$p \Rightarrow (p \Rightarrow q)$
T	T	T	T
T	F	F	F
F	T	T	T
F	F	T	T

Since entries in the last column of the truth table depend on statements p, q and $p \Rightarrow q$, therefore, given statement is a contingency.

(b) Truth table for the given statement is shown in Table 12.57.

Table 12.57 $(p \wedge \sim q) \vee (\sim p \wedge q)$

p	q	$\sim p$	$\sim q$	$p \wedge \sim q$	$\sim p \wedge q$	$(p \wedge \sim q) \vee (\sim p \wedge q)$
T	T	F	F	F	F	F
T	F	F	T	T	F	T
F	T	T	F	F	T	T
F	F	T	T	F	F	F

Since entries in the last column of the truth table depend on component statements associated with this compound statement, therefore, it is a contingency.

(c) Truth table for the given statement is shown in Table 12.58.

Table 12.58 $\sim (p \wedge q) \vee (\sim p \wedge \sim q)$

p	q	$\sim p$	$\sim q$	$p \wedge q$	$\sim(p \wedge q)$	$\sim p \vee \sim q$	$\sim (p \wedge q) \vee (\sim p \wedge \sim q)$
T	T	F	F	T	F	F	F
T	F	F	T	F	T	T	T
F	T	T	F	F	T	T	T
F	F	T	T	F	T	T	T

Since entries in the last column of the truth table depend on component statements associated with this compound statement, therefore, it is a contingency.

(d) Truth table for the given statement is shown in Table 12.59.

Name	Rule of Inference	Tautological Form
Addition	$\dfrac{p}{\therefore p \vee q}$	$p \Rightarrow p \vee q$
Simplification	$\dfrac{p \wedge q}{\therefore p}$	$(p \wedge q) \Rightarrow p$
Conjunction	$\begin{array}{c} p \\ \underline{q} \\ \therefore p \wedge q \end{array}$	$[(p) \wedge (q)] \Rightarrow (p \wedge q)$
Modus ponens	$\begin{array}{c} p \Rightarrow q \\ \underline{p} \\ \therefore q \end{array}$	$[(p \Rightarrow q) \wedge p] \Rightarrow q$
Modus tollens	$\begin{array}{c} p \Rightarrow q \\ \underline{\sim q} \\ \therefore \sim p \end{array}$	$[(p \Rightarrow q) \wedge \sim q] \Rightarrow \sim p$
Hypothatical syllogism	$\dfrac{p \Rightarrow q \text{ and } q \Rightarrow r}{\therefore P \Rightarrow r}$	$[(p \Rightarrow q) \wedge (q \Rightarrow r)] \Rightarrow (p \Rightarrow r)$
Constructive dilemma	$\begin{array}{c} (p \Rightarrow q) \wedge (r \Rightarrow s) \\ \underline{p \wedge r} \\ \therefore q \vee s \end{array}$	$[(p \Rightarrow q) \wedge (r \Rightarrow s) \wedge (p \wedge r)] \Rightarrow (q \vee s)$
Disjunction dilemma	$\begin{array}{c} p \wedge q \\ \underline{\sim p} \\ \therefore q \end{array}$	$[(p \wedge q) \wedge (\sim p)] \Rightarrow q$
Destructive dilemma	$\begin{array}{c} p \Rightarrow q \wedge (r \Rightarrow s) \\ \underline{\sim q \vee \sim s} \\ \therefore \sim p \vee \sim r \end{array}$	$[(p \Rightarrow q) \wedge (r \Rightarrow s) \wedge (\sim q \vee \sim s)] \Rightarrow (\sim p \vee \sim r)$
Absorption	$\dfrac{p \Rightarrow q}{\therefore p \Rightarrow (p \wedge q)}$	$[(p \Rightarrow q) \Rightarrow (p \Rightarrow (p \wedge q))]$