...(n) is true for ...   ...mathematical induction

Example 10 By the principle of induction, show that $3^{4n+2} + 5^{2n+1}$ is a multiple of 14, for all positive integral value of n including zero.

Solution: Let the given expression P (n) : $3^{4n+2} + 5^{2n+1}$ be the multiple of 14.

Basis of induction: If $n = 1$, then we have P (1) : $3^{4\cdot1+2} + 5^{2\cdot1+1} = 854 = 14 \times 61$ which is a multiple of 14. Similarly, for $n = 2$,

$$P(2) : 3^{4\cdot2+2} + 5^{2\cdot2+1} = 62174 = 14 \times 4441$$

a multiple of 14.

Induction step: Assuming that the result is true for $n = k$, then

$$P(k) = 3^{4k+2} + 5^{2k+1} = 14 \times t \,;\, t \in \mathrm{I}$$

multiple of 14.

Replacing k by $k + 1$ in P (k), we get

$$
\begin{aligned}
3^{4(k+1)+2} + 5^{2(k+1)+1} &= 3^{4k+6} + 5^{2k+3} \\
&= 3^{4k+2} . 3^4 + 5^{2k+1} . 5^2 \\
&= 3^{4k+2}\,(11 + 70) + 5^{2k+1}\,(11 + 14) \\
&= 11(3^{4k+2} + 5^{2k+1}) + 70 \cdot 3^{4k+2} + 14 \cdot 5^{2k+1} \\
&= 11 \cdot 14\, t + 14\{5 \cdot 3^{4k+2} + 5^{2k+1}\} \\
&= 14\{11\, t + 5 \cdot 3^{4k+2} + 5^{2k+1}\}
\end{aligned}
$$

which is a multiple of 14. Hence, the result is true for $n = k + 1$.

Moreover, for $n = 0$, we have

$$\text{P}\,(0) : 3^{4 \cdot 0 + 2} + 5^{2 \cdot 0 + 1} = 14 \cdot 1$$

which is a multiple of 14. Hence, P (n) also holds true for $n = 0$.

Example 11 (*a*) If nth term of A.P. is $a + (n - 1)\, d$, then show by the principle of mathematical induction that the sum of n terms of A.P. is $\frac{n}{2}\{2a + (n - 1)\, d\}$. That is, by the principle of mathematical induction, prove that

$$\text{P}\,(n) : a + (a + d) + (a + 2d) + \ldots + \{a + (n - 1)\, d\} = \frac{n}{2}\{2a + (n - 1)\, d\}$$

(*b*) Prove by the principle of mathematical induction the result

$$\text{P}\,(n) : a + ar + ar^2 + \ldots + ar^{n-1} = a \cdot \frac{r^n - 1}{r - 1}, \text{ if } r \neq 1$$

VV9 **Example 14** Prove by the principle of mathematical induction that P (n): $10^n + 3 \cdot 4^{n+2} + 5$ is divisible by 9.

Solution *Basis of induction:* For $n = 1$, we have

$$10^1 + 3 \cdot 4^{1+2} + 5 = 207 = 9 \times 23$$

This is divisible by 9.

Induction step: Assuming that P (n) is true for $n = k$, so that $10^k + 3 \cdot 4^{k+2} + 5$ is divisible by 9.

Replacing k by $k + 1$ in P (k), we get

$$\begin{aligned} 10^{k+1} + 3 \cdot 4^{k+1+2} + 5 &= 10 \cdot 10^k + 3 \cdot 4 \cdot 4^{k+2} + 5 \\ &= (9 + 1) \cdot 10^k + 3\,(3 + 1) \cdot 4^{k+2} + 5 \\ &= 9 \cdot 10^k + 10^k + 3 \cdot 3 \cdot 4^{k+2} + 3 \cdot 4^{k+2} + 5 \\ &= 9 \cdot 10^k + 9 \cdot 4^{k+2} + (10^k + 3 \cdot 4^{k+2} + 5) \\ &= 9 \cdot 10^k + 9 \cdot 4^{k+2} + 9m \qquad (\text{where } 9m = 10^k + 3.4^{k+2} + 5) \\ &= 9(10^k + 4^{k+2} + m) = 9t \;;\ \text{for } t = 10^k + 4^{k+2} + m \end{aligned}$$

which is divisible by 9.

This shows that if P (n) is true for $n = k$, then it is also true for $n = k + 1$. Hence, P (n) is true for all positive integral values of n.

Example 15 Prove by the principle of mathematical induction

$$\text{P}(n) : 1^3 + 2^3 + 3^3 + \ldots + n^3 = \frac{n^2 (n+1)^2}{4}$$

Solution *Basis of induction:* For $n = 1$, the LHS of P (n) is $1^3 = 1$ and RHS is also $\frac{1^2 (1+1)^2}{4} = 1$. Hence, P (n) is true for $n = 1$.

Induction step: Assuming that P (n) is true for $n = k$. Then we get

$$\text{P}(k) : 1^3 + 2^3 + 3^3 + \ldots + k^3 = \frac{k^2 (k+1)^2}{4} \qquad \text{...(ii)}$$

Adding the term $(k + 1)^3$ to both sides of P (k), we get

$$\begin{aligned} 1^3 + 2^3 + 3^3 + \ldots + k^3 + (k + 1)^3 &= \frac{k^2 (k+1)^2}{4} + (k + 1)^3 \\ &= \frac{(k+1)^2}{4} \{k^2 + 4\,(k + 1)\} \\ &= \frac{(k+1)^2 \,(k+2)^2}{4} = \frac{(k+1)^2 \,\{(k+1)+1\}^2}{4} \end{aligned}$$

This shows that if P (n) is true for $n = k$, then it is also true for $n = k + 1$. Hence, by mathematical induction P (n) is true for every positive integral value of n.